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# The global attractivity of the rational difference equation

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}}$$

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## Abstract

This paper studies global asymptotic stability for positive solutions to the equation

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}}, \quad n = 0, 1, \dots,$$

with  $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, \infty)$  and  $1 \leq k < m$ . The paper includes a discussion of stability for a wide class of symmetric rational difference equations which includes the type studied here as well as several other in the recent literature.

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## 1. Introduction

This paper studies the behavior of positive solutions of the recursive equation

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}}, \quad n = 0, 1, \dots, \tag{1}$$

with  $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, \infty)$  and  $1 \leq k < m$ .

The study of properties of rational difference equations has been an area of intense interest in recent years; see [1,2] and the references therein.

Here we prove the following result for higher order rational equations.

**Theorem 1.** *Suppose that  $\{y_i\}$  satisfies (1) with  $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, \infty)$ . Then, the sequence  $\{y_i\}$  converges to the unique equilibrium 1.*

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Investigation of Eq. (1) is motivated by several recent results. In Li and Zhu [3], qualities of equations of the form

$$x_n = \frac{1 + x_{n-1}x_{n-3}}{x_{n-1} + x_{n-3}}, \quad n \in \mathbb{N}_0 \quad (2)$$

and

$$x_n = \frac{1 + x_{n-2}x_{n-3}}{x_{n-2} + x_{n-3}} \quad n \in \mathbb{N}_0, \quad (3)$$

were studied. In addition, in [4,5], Li investigates the qualitative behavior of the equations

$$x_n = \frac{x_{n-1}x_{n-2}x_{n-4} + x_{n-1} + x_{n-2} + x_{n-4}}{x_{n-1}x_{n-2} + x_{n-1}x_{n-4} + x_{n-2}x_{n-4} + 1}, \quad n \in \mathbb{N}_0 \quad (4)$$

and

$$x_n = \frac{x_{n-2}x_{n-3}x_{n-4} + x_{n-2} + x_{n-3} + x_{n-4}}{x_{n-2}x_{n-3} + x_{n-2}x_{n-4} + x_{n-3}x_{n-4} + 1} \quad n \in \mathbb{N}_0. \quad (5)$$

The striking similarities among these equations suggest investigation of stability properties over wide ranges of equations defined through symmetric rational functions. Here we demonstrate that the transformation method introduced in a recent paper of the authors [6] can be very useful for such investigations. We remark that stability for equations of the form

$$y_n = \frac{1 + y_{n-k}y_{n-m}}{y_{n-k} + y_{n-m}}, \quad n = 0, 1, \dots,$$

as in (2) and (3) (see also [7] for the case  $k = 1$  and  $m = 2$ ) can also be shown via almost identical calculations.

For further recent studies of rational difference equations of lower order see [8–10].

**Remark.** It is worthwhile to note at this point that global asymptotic stability for the special cases in Eqs. (2) and (3) is proved in [3] via analysis of semi-cycle structure (similar methods are also used in [4,5]). Such analysis, while computationally feasible for small  $m$  and  $k$ , can be very involved for larger values. In fact, determination of semi-cycle structure as a function of  $(m, k)$  appears to be an interesting algebraic/number theoretic problem in its own right. It can be verified that, for  $k = 1$  and  $m = 5$ , there are four possible cycle structures for solutions to (6): one of period 21, one of period 7, one of period 3, and one of period 1. It is fortunate that the transformation method used here does not require prior determination of detailed semi-cycle structure.

The paper proceeds as follows. In Section 2, we introduce some preliminary lemmas and notation. Section 3 contains a proof of Theorem 1, while in Section 4 we discuss stability for a wide class of symmetric rational difference equations which include those mentioned above.

## 2. Preliminaries and notation

In this section, we introduce some preliminary lemmas and notation.

First, consider the simple transformed sequence  $\{y_i^*\}$  defined by

$$y_i^* = \begin{cases} y_i, & \text{if } y_i \geq 1 \\ \frac{1}{y_i}, & \text{otherwise.} \end{cases} \quad (6)$$

The following elementary lemma will be useful.

**Lemma 1.** Suppose  $f$  is defined by

$$f(x, y) = \frac{x + y}{1 + xy}. \quad (7)$$

Then,  $f$  is decreasing in  $x$  if and only if  $y > 1$  and increasing in  $x$  if and only if  $y < 1$ .

**Proof.** This follows directly from the fact that

$$\frac{\partial}{\partial x} f(x, y) = \frac{1 - y^2}{(1 + xy)^2}. \quad \square \quad (8)$$

Next we prove the following contraction lemma (similar to Lemma 1 in [6]) which will be helpful in showing convergence of solutions in the transformed space obtained through (6).

**Lemma 2.** *We have*

$$1 \leq y_n^* \leq \max\{y_{n-k}^*, y_{n-m}^*\}, \quad (9)$$

for all  $n \geq m$ .

**Proof.** First, note that

$$y_{n-k} + y_{n-m} - (1 + y_{n-k}y_{n-m}) = -(1 - y_{n-k})(1 - y_{n-m}) \quad (10)$$

and hence from (1),  $y_n \leq 1$  whenever  $y_{n-k} - 1$  and  $y_{n-m} - 1$  are of the same signs, and  $y_n \geq 1$  otherwise. Let  $x = \max\{y_{n-k}^*, y_{n-m}^*\} \geq 1$ . We have four cases to consider.

**Case (i)** ( $y_{n-k} \leq 1$ ,  $y_{n-m} \geq 1$ ,  $y_n \geq 1$ ). Here, by (8) and (6), we have

$$y_n^* \leq \frac{x + 1/x}{2} \leq x. \quad (11)$$

**Case (ii)** ( $y_{n-k} \geq 1$ ,  $y_{n-m} \leq 1$ ,  $y_n \geq 1$ ). The argument is identical to that in Case (i).

**Case (iii)** ( $y_{n-k} \leq 1$ ,  $y_{n-m} \leq 1$ ,  $y_n \leq 1$ ). Here,

$$y_n^* = \frac{1 + y_{n-k}y_{n-m}}{y_{n-k} + y_{n-m}} \leq \frac{1 + 1/x^2}{2/x} = \frac{x + 1/x}{2} \leq x. \quad (12)$$

**Case (iv)** ( $y_{n-k} \geq 1$ ,  $y_{n-m} \geq 1$ ,  $y_n \leq 1$ ). Here,

$$y_n^* = \frac{1 + y_{n-k}y_{n-m}}{y_{n-k} + y_{n-m}} \leq \frac{1 + x^2}{2x} = \frac{1/x + x}{2} \leq x. \quad \square \quad (13)$$

Now, set

$$D_n = \max_{n-m \leq i \leq n-1} \{y_i^*\}, \quad (14)$$

for  $n \geq m$ .

The following lemma is a simple consequence of Lemma 2 and (14).

**Lemma 3.** *The sequence  $\{D_i\}$  is monotonically non-increasing in  $i$ , for  $i \geq m$ .*

Since  $D_i \geq 1$  for  $i \geq m$ , Lemma 3 implies that, as  $i$  tends to infinity, the sequence  $\{D_i\}$  converges to some limit, say  $D$ , where  $D \geq 1$ .

We now turn to a proof of Theorem 1.

### 3. Convergence of solutions to Eq. (1)

In this section, we prove Theorem 1.

**Proof of Theorem 1.** Note that it suffices to show that the transformed sequence  $\{y_i^*\}$  converges to 1.

By the definition in (14), the values of  $D_i$  are taken on by entries in the sequence  $\{y_j^*\}$ , and as well, by Lemma 2,  $y_i^* \in [1, D_i]$  for  $i \geq m$ . Suppose  $D > 1$ . Then, for any  $\epsilon \in (0, D)$ , we can find an  $N$  such that  $y_N^* \in [D, D + \epsilon]$ , and for  $i \geq N - m$ ,

$$y_i^* \in [1, D + \epsilon]. \quad (15)$$

We again consider the four possible cases, and show that  $D = 1$ . From this, (6), (14) and the definition of  $D$ , the result follows.

**Case (i)** ( $y_{n-k} \leq 1$ ,  $y_{n-m} \geq 1$ ,  $y_n \geq 1$ ). Here, by (8) and (6), we have

$$D \leq y_n^* = y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}} \leq \frac{1/(D + \epsilon) + D + \epsilon}{2}. \quad (16)$$

Hence

$$\begin{aligned} 2D(D + \epsilon) &\leq (D + \epsilon)^2 + 1 \\ 2D^2 + 2D\epsilon &\leq D^2 + 2D\epsilon + \epsilon^2 + 1 \\ D^2 &\leq 1 + \epsilon^2, \end{aligned} \quad (17)$$

which implies  $D = 1$ , since  $\epsilon > 0$  is arbitrary.

**Case (ii)** ( $y_{n-k} \geq 1$ ,  $y_{n-m} \leq 1$ ,  $y_n \geq 1$ ). The argument is identical to that in Case (i).

**Case (iii)** ( $y_{n-k} \leq 1$ ,  $y_{n-m} \leq 1$ ,  $y_n \leq 1$ ). Here,

$$D \leq y_n^* = \frac{1}{y_n} \leq \frac{1 + 1/(D + \epsilon)^2}{2/(D + \epsilon)} = \frac{D + \epsilon + 1/(D + \epsilon)}{2}, \quad (18)$$

and the argument continues as in Case (i).

**Case (iv)** ( $y_{n-k} \geq 1$ ,  $y_{n-m} \geq 1$ ,  $y_n \leq 1$ ). Here,

$$D \leq y_n^* = \frac{1}{y_n} \leq \frac{1 + (D + \epsilon)^2}{2(D + \epsilon)}, \quad (19)$$

and the argument follows as before.  $\square$

In the next section, we consider briefly equations defined through more general symmetric rational functions.

#### 4. Stability for more general rational symmetric functions

As mentioned earlier, the global asymptotic stability of positive solutions to the various equations listed above suggests that the same potentially holds for similar rational symmetric functions. We conjecture the following natural generalization of Eqs. (4) and (5).

**Conjecture 1.** Suppose that  $\{y_i\}$  satisfies

$$y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1}, \quad n \in \mathbb{N}_0 \quad (20)$$

with  $y_{-m}, y_{-m+1}, \dots, y_{-1} \in (0, \infty)$  and  $1 \leq k < l < m$ . Then, the sequence  $\{y_i\}$  converges to the unique equilibrium 1.

More generally, we have

**Conjecture 2.** Suppose that  $v$  is odd and  $1 \leq k_1 < k_2 < \dots < k_v$ , and define  $S = \{1, 2, \dots, v\}$ . If  $\{y_i\}$  satisfies

$$y_n = \frac{f_1(y_{n-k_1}, y_{n-k_2}, \dots, y_{n-k_v})}{f_2(y_{n-k_1}, y_{n-k_2}, \dots, y_{n-k_v})}, \quad n \in \mathbb{N}_0 \quad (21)$$

where

$$f_1(x_1, x_2, \dots, x_v) = \sum_{\substack{r=1 \\ r \text{ odd}}}^v \sum_{\substack{\{t_1, t_2, \dots, t_r\} \subset S \\ t_1 < t_2 < \dots < t_r}} x_{t_1} x_{t_2} \dots x_{t_r}, \quad (22)$$

and

$$f_2(x_1, x_2, \dots, x_v) = 1 + \sum_{\substack{r=2 \\ r \text{ even}}}^{v-1} \sum_{\substack{\{t_1, t_2, \dots, t_r\} \subset S \\ t_1 < t_2 < \dots < t_r}} x_{t_1} x_{t_2} \dots x_{t_r}, \quad (23)$$

with  $y_{-k_v}, y_{-k_v+1}, \dots, y_{-1} \in (0, \infty)$ , then the sequence  $\{y_i\}$  converges to the unique equilibrium 1.

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